This example is set up in ECF Webster (with bold TX fonts). It uses:

```
\usepackage{txfonts}
\usepackage[upright]{txgreeks}
\renewcommand\familydefault{fwb} % emerald package
\usepackage{mathastext}
\renewcommand{\int}{\intop\limits}
\linespread{1.5}
\begin{document}\mathversion{bold}
```

Typeset with mathastext 1.15c (2012/10/05).

To illustrate some Hilbert Space properties of the co-Poisson summation, we will assume $K = Q$. The components (a_v) of an adele a are written a_p at finite places and a_r at the real place. We have an embedding of the Schwartz space of test-functions on R into the Bruhat-Schwartz space on A which sends $\psi(x)$ to $\varphi(a) = \prod_p 1_{|a_p|_p \leq 1}(a_p) \cdot \psi(a_r)$, and we write $E'_R(g)$ for the distribution on R thus obtained from $E'(g)$ on A.

Theorem 1. Let g be a compact Bruhat-Schwartz function on the ideles of Q. The co-Poisson summation $E'_R(g)$ is a square-integrable function (with respect to the Lebesgue measure). The $L^2(R)$ function $E'_R(g)$ is equal to the constant $-\int_{A^\times} g(v)|v|^{-1/2}d^*v$ in a neighborhood of the origin.

Proof. We may first, without changing anything to $E'_R(g)$, replace g with its average under the action of the finite unit ideles, so that it may be assumed invariant. Any such compact invariant g is a finite linear combination of suitable multiplicative translates of functions of the type $g(v) = \prod_p 1_{|v_p|_p=1}(v_p) \cdot f(v_r)$ with $f(t)$ a smooth compactly supported function on $R^\times$, so that we may assume that g has this form. We claim that:

$$\int_{A^\times} |\varphi(v)| \sum_{q \in Q^\times} |g(qv)| \sqrt{|v|}\, d^*v < \infty$$

Indeed $\sum_{q \in Q^\times} |g(qv)| = |f(|v|)| + |f(-|v|)|$ is bounded above by a multiple of $|v|$. And $\int_{A^\times} |\varphi(v)||v|^{3/2}\, d^*v < \infty$ for each

Bruhat-Schwartz function on the adeles (basically, from $\prod_p (1-p^{-3/2})^{-1} < \infty$). So

$$E'(g)(\varphi) = \sum_{q\in Q^\times} \int_{A^\times} \varphi(v)g(qv)\sqrt{|v|}\, d^*v - \int_{A^\times} \frac{g(v)}{\sqrt{|v|}} d^*v \int_A \varphi(x)\, dx$$

$$E'(g)(\varphi) = \sum_{q\in Q^\times} \int_{A^\times} \varphi(v/q)g(v)\sqrt{|v|}\, d^*v - \int_{A^\times} \frac{g(v)}{\sqrt{|v|}} d^*v \int_A \varphi(x)\, dx$$

Let us now specialize to $\varphi(a) = \prod_p 1_{|a_p|_p \le 1}(a_p) \cdot \psi(a_r)$. Each integral can be evaluated as an infinite product. The finite places contribute 0 or 1 according to whether $q \in Q^\times$ satisfies $|q|_p < 1$ or not. So only the inverse integers $q = 1/n$, $n \in Z$, contribute:

$$E'_R(g)(\psi) = \sum_{n\in Z^\times} \int_{R^\times} \psi(nt)f(t)\sqrt{|t|}\frac{dt}{2|t|} - \int_{R^\times} \frac{f(t)}{\sqrt{|t|}}\frac{dt}{2|t|} \int_R \psi(x)\, dx$$

We can now revert the steps, but this time on $R^\times$ and we get:

$$E'_R(g)(\psi) = \int_{R^\times} \psi(t) \sum_{n\in Z^\times} \frac{f(t/n)}{\sqrt{|n|}} \frac{dt}{2\sqrt{|t|}} - \int_{R^\times} \frac{f(t)}{\sqrt{|t|}}\frac{dt}{2|t|} \int_R \psi(x)\, dx$$

Let us express this in terms of $\alpha(y) = (f(y)+f(-y))/2\sqrt{|y|}$:

$$E'_R(g)(\psi) = \int_R \psi(y) \sum_{n\ge 1} \frac{\alpha(y/n)}{n} dy - \int_0^\infty \frac{\alpha(y)}{y} dy \int_R \psi(x)\, dx$$

So the distribution $E'_R(g)$ is in fact the even smooth function

$$E'_R(g)(y) = \sum_{n\ge 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y} dy$$

As $\alpha(y)$ has compact support in $R\setminus\{0\}$, the summation over $n \geq 1$ contains only vanishing terms for $|y|$ small enough. So $E'_R(g)$ is equal to the constant $-\int_0^\infty \frac{\alpha(y)}{y}dy = -\int_{R^\times} \frac{f(y)}{\sqrt{|y|}}\frac{dy}{2|y|} = -\int_{A^\times} g(t)/\sqrt{|t|}\,d^*t$ in a neighborhood of 0. To prove that it is L^2, let $\beta(y)$ be the smooth compactly supported function $\alpha(1/y)/2|y|$ of $y \in R$ ($\beta(0) = 0$). Then ($y \neq 0$):

$$E'_R(g)(y) = \sum_{n\in Z} \frac{1}{|y|}\beta(\frac{n}{y}) - \int_R \beta(y)\,dy$$

From the usual Poisson summation formula, this is also:

$$\sum_{n\in Z} \gamma(ny) - \int_R \beta(y)\,dy = \sum_{n\neq 0} \gamma(ny)$$

where $\gamma(y) = \int_R \exp(i\,2\pi yw)\beta(w)\,dw$ is a Schwartz rapidly decreasing function. From this formula we deduce easily that $E'_R(g)(y)$ is itself in the Schwartz class of rapidly decreasing functions, and in particular it is is square-integrable. □

It is useful to recapitulate some of the results arising in this proof:

Theorem 2. Let g be a compact Bruhat-Schwartz function on the ideles of Q. The co-Poisson summation $E'_R(g)$ is an even function on R in the Schwartz class of rapidly decreasing functions. It is constant, as well as its Fourier Transform, in a neighborhood of the origin. It may be written as

$$E'_R(g)(y) = \sum_{n\geq 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y} dy$$

with a function $\alpha(y)$ smooth with compact support away from the origin, and conversely each such formula corresponds to the co-Poisson summation $E'_R(g)$ of a compact Bruhat-Schwartz function on the ideles of Q. The Fourier transform $\int_R E'_R(g)(y) \exp(i2\pi wy)\, dy$ corresponds in the formula above to the replacement $\alpha(y) \mapsto \alpha(1/y)/|y|$.

Everything has been obtained previously.