# Qu’est-ce que c’est ?

$1500\, s_6 s_5^4 s_2^2 - 22500\, s_6 s_5^4 s_4 + 62208\, s_4 s_2 s_6^4 - 46656\, s_6^5 + 3125\, s_5^6 - 900\, s_4 s_2^3 s_5^4 + 2000\, s_4^2 s_2 s_5^4 + 9216\, s_4^4 s_2 s_6^2 +$
$9216\, s_4 s_2^4 s_6^3 - 64\, s_4^4 s_2^4 s_6 - 17280\, s_4^2 s_2^2 s_6^3 - 4352\, s_4^3 s_2^3 s_6^2 + 16\, s_4^3 s_2^4 s_5^2 + 512\, s_4^5 s_2^2 s_6 + 512\, s_4^2 s_2^5 s_6^2 - 32400\, s_4 s_1^2 s_6^4 -$
$50\, s_4^2 s_1^2 s_5^4 - 192\, s_4^4 s_1^2 s_6^2 + 1500\, s_4^2 s_1^4 s_6^3 - 36\, s_4^3 s_1^3 s_5^3 - 27\, s_4^4 s_1^4 s_5^2 + 108\, s_4^5 s_1^4 s_6 - 1350\, s_3^3 s_1^3 s_6^3 + 38880\, s_6^4 s_1 s_5 +$
$540\, s_6^3 s_1^2 s_5^2 - 32400\, s_6^3 s_5^2 s_2 + 27000\, s_3 s_1^3 s_6^4 + 410\, s_6^2 s_1^3 s_5^3 + 27000\, s_6^2 s_5^3 s_3 + 3888\, s_4 s_2 s_3^2 s_6^3 - 4860\, s_4 s_2 s_3^4 s_6^2 - 630\, s_4 s_2 s_3^3 s_5^3 -$
$192\, s_4 s_2 s_1^4 s_5^4 - 4\, s_4^3 s_2^3 s_1^2 s_5^2 - 6\, s_4^2 s_2^2 s_1^3 s_5^3 + 560\, s_4^2 s_2^2 s_5^3 s_3 + 4816\, s_4^3 s_2^2 s_1^2 s_6^2 + 24\, s_4^2 s_2^3 s_1 s_5^3 + 4816\, s_4^2 s_2^3 s_6 s_5^2 + 16\, s_4^4 s_2^3 s_1^2 s_6 -$
$6480\, s_4 s_2^2 s_6^2 s_5^2 + 1020\, s_4 s_2^2 s_1^2 s_5^4 + 8208\, s_4^2 s_2^2 s_3^2 s_6^2 - 72\, s_4 s_2^4 s_3 s_5^3 - 576\, s_4 s_2^4 s_3^2 s_6^2 + 2808\, s_4 s_2^2 s_3^3 s_1 s_6^2 - 6480\, s_4^2 s_2 s_1^2 s_6^3 +$
$34992\, s_3^2 s_6^4 - 8748\, s_3^4 s_6^3 - 13824\, s_4^3 s_6^3 - 13824\, s_2^3 s_6^4 + 256\, s_1^5 s_5^5 + 2808\, s_4^2 s_2 s_3^3 s_5 s_6 - 4536\, s_4^2 s_2 s_3^2 s_1^2 s_6^2 - 22896\, s_4 s_2 s_3^2 s_6^2 s_1 s_5 +$
$356\, s_4 s_2^2 s_3^2 s_1 s_5^3 - 4536\, s_4 s_2^2 s_3^2 s_6 s_5^2 + 144\, s_4 s_2^3 s_3^2 s_1^2 s_6^2 - 3456\, s_4 s_2^2 s_3 s_1 s_6^3 + 18\, s_4 s_2^3 s_3 s_1^2 s_5^3 - 13040\, s_4^2 s_2 s_1^3 s_5 s_6^2 -$
$5760\, s_4^3 s_2 s_1 s_3 s_6^2 - 5760\, s_4 s_2^3 s_3 s_5 s_6^2 - 3456\, s_4^2 s_2 s_5 s_3 s_6^2 + 1020\, s_4^2 s_2 s_1^4 s_5^2 s_6 - 2050\, s_4 s_2 s_1^4 s_5 s_3 s_6^2 - 746\, s_4^2 s_2 s_1^2 s_5^3 s_3 -$
$13040\, s_4 s_2^2 s_6 s_1 s_5^3 - 80\, s_4 s_2 s_3^2 s_1^3 s_5^3 - 630\, s_4 s_2 s_3^3 s_1^3 s_6^2 + 31968\, s_4 s_2 s_6^3 s_1 s_5 + 8748\, s_4 s_2 s_6^2 s_1^2 s_5^2 + 19800\, s_4 s_2 s_3 s_1^3 s_6^3 -$
$2050\, s_4 s_2 s_1 s_5^4 s_3 - 1584\, s_4^2 s_2^2 s_3^2 s_6 s_1 s_5 - 2496\, s_4^2 s_2^3 s_3 s_1 s_6^2 + 24\, s_4^3 s_2^2 s_1^3 s_5 s_6 + 320\, s_4^4 s_2^2 s_1 s_3 s_6 + 320\, s_4^2 s_2^4 s_3 s_5 s_6 -$
$2496\, s_4^3 s_2^2 s_5 s_3 s_6 + 15264\, s_4^2 s_2^2 s_6^2 s_1 s_5 - 5428\, s_4^2 s_2^2 s_6 s_1^2 s_5^2 + 560\, s_4^2 s_2^2 s_3 s_1^3 s_6^2 - 96\, s_4^3 s_2^3 s_6 s_1 s_5 - 80\, s_4^2 s_2^3 s_3 s_1^2 s_6 s_5 -$
$80\, s_4^3 s_2^2 s_1 s_5^2 s_3 + 356\, s_4^2 s_2 s_3^2 s_1^3 s_5 s_6 + 10152\, s_4 s_2^2 s_3 s_1^2 s_6^2 s_5 - 746\, s_4 s_2^2 s_3 s_1^3 s_6 s_5^2 + 3272\, s_4 s_2^3 s_3 s_1 s_6 s_5^2 + 3272\, s_4^3 s_2 s_1^2 s_5 s_3 s_6 +$
$9768\, s_4 s_2 s_6 s_1^3 s_5^3 + 19800\, s_4 s_2 s_6 s_5^3 s_3 - 10560\, s_4 s_2^3 s_1^2 s_6^3 + 160\, s_4^3 s_2 s_1 s_5^3 - 10560\, s_4^3 s_2 s_5^2 s_6 - 900\, s_4^3 s_2 s_1^4 s_6^2 + 144\, s_4^4 s_2 s_1^2 s_5^2 -$
$576\, s_4^5 s_2 s_1^2 s_6 - 576\, s_4^4 s_2 s_3^2 s_6 + 144\, s_4^3 s_2 s_3^2 s_5^2 - 576\, s_4 s_2^5 s_5^2 s_6 + 2000\, s_4 s_2^2 s_1^4 s_6^3 - 128\, s_4^2 s_2^4 s_1^2 s_6^2 - 27540\, s_4 s_1^2 s_3^2 s_6^3 +$
$162\, s_4 s_1^2 s_3^4 s_6^2 + 24\, s_4 s_1^2 s_3^3 s_5^3 + 825\, s_4^2 s_1^4 s_3^2 s_6^2 + 2250\, s_4^2 s_1^5 s_5 s_6^2 - 120\, s_4^3 s_1^3 s_3 s_6^2 + 144\, s_4^2 s_1^4 s_5^3 s_3 - 1800\, s_4 s_1^3 s_6^3 s_5 -$
$1700\, s_4 s_1^4 s_6^2 s_5^2 - 3750\, s_4 s_1^5 s_3 s_6^3 + 160\, s_4 s_1^3 s_5^4 s_3 - 1600\, s_4 s_1^5 s_6 s_5^3 + 248\, s_4^3 s_1^2 s_5^2 s_6 + 24\, s_4^4 s_1^2 s_3^2 s_6 - 6\, s_4^3 s_1^2 s_3^2 s_5^2 +$
$144\, s_4^4 s_1^3 s_5 s_6 + 21384\, s_3^3 s_1 s_2 s_6^3 + 21384\, s_3^3 s_5 s_4 s_6^2 + 15552\, s_3^2 s_6^3 s_1 s_5 - 27540\, s_3^2 s_6^2 s_5^2 s_2 - 9720\, s_3^2 s_1^2 s_2^2 s_6^3 - 77760\, s_3 s_1 s_2 s_6^4 +$
$46656\, s_1 s_4^2 s_3 s_6^3 + 46656\, s_3 s_2^2 s_5 s_6^3 - 77760\, s_5 s_4 s_3 s_6^3 + 2250\, s_1^4 s_5 s_3 s_6^3 - 1800\, s_6^2 s_1 s_5^3 s_2 + 248\, s_1^2 s_2^3 s_5^2 s_6^2 - 21888\, s_1 s_2^3 s_5 s_6^3 +$
$15600\, s_1^3 s_2^2 s_6^3 s_5 - 21888\, s_5 s_1 s_4^3 s_6^2 - 128\, s_4^4 s_2^2 s_5^2 + 10152\, s_4^2 s_2 s_6 s_1 s_5^2 s_3 + 144\, s_4 s_2^4 s_1^2 s_5^2 s_6 - 640\, s_4 s_2^4 s_1 s_5 s_6^2 + 160\, s_4 s_2^3 s_1^3 s_6^2 s_5 -$
$72\, s_4^4 s_2 s_1^3 s_3 s_6 + 18\, s_4^3 s_2 s_1^3 s_3 s_5^2 - 640\, s_4^4 s_2 s_5 s_1 s_6 - 108\, s_4^2 s_1^2 s_3^3 s_5 s_6 + 1980\, s_4 s_1^3 s_3^2 s_6^2 s_5 - 2412\, s_4 s_1^2 s_3^2 s_6 s_5^2 s_2 + 16632\, s_4^2 s_1^2 s_5 s_3 s_6^2 -$
$630\, s_4^3 s_1^4 s_5 s_3 s_6 - 12330\, s_4 s_1^2 s_6 s_5^3 s_3 - 682\, s_4^2 s_1^3 s_6 s_5^2 s_3 - 31320\, s_3 s_1^2 s_2 s_6^3 s_5 - 12330\, s_3 s_1^3 s_2 s_6^2 s_5^2 + 16632\, s_3 s_1 s_2^2 s_6^2 s_5^2 -$
$8640\, s_3^2 s_2^3 s_6^3 + 43200\, s_1^2 s_2^2 s_6^4 + 43200\, s_5^2 s_4^2 s_6^2 - 8640\, s_4^3 s_3^2 s_6^2 - 192\, s_2^4 s_5^2 s_6^2 - 22500\, s_2 s_1^4 s_6^4 - 900\, s_1 s_5^4 s_3^3 + 2250\, s_1 s_5^5 s_2^2 -$
$2500\, s_1 s_5^5 s_4 - 128\, s_1^4 s_5^4 s_3^2 + 2000\, s_1^2 s_5^5 s_3 - 1600\, s_1^3 s_5^5 s_2 - 1350\, s_6 s_3^3 s_5^3 + 320\, s_6 s_1^4 s_5^4 - 31320\, s_6^2 s_1 s_4 s_5^2 s_3 - 6318\, s_1 s_5 s_3^4 s_6^2 +$
$15417\, s_1^2 s_5^2 s_3^2 s_6^2 + 560\, s_1^2 s_5^4 s_3^2 s_2 + 144\, s_1^3 s_5^4 s_3 s_2^2 + 2000\, s_1^5 s_5^2 s_3 s_6^2 - 900\, s_1^4 s_5 s_3^3 s_6^2 - 630\, s_1 s_5^4 s_3 s_2^3 + 1020\, s_1 s_5^3 s_4^2 s_3^2 +$
$144\, s_1 s_5^3 s_2^4 s_6 - 2500\, s_1^5 s_5 s_2 s_6^3 - 36\, s_1^3 s_5^3 s_2^3 s_6 - 50\, s_1^4 s_5^2 s_2^2 s_6^2 - 208\, s_6 s_3^2 s_1^3 s_5^3 + 2250\, s_6 s_1 s_5^4 s_3 - 1700\, s_6 s_1^2 s_5^4 s_2 -$
$120\, s_6 s_3 s_2^3 s_5^3 + 15600\, s_6 s_1 s_4^2 s_5^3 - 9720\, s_6 s_4^2 s_3^2 s_5^2 + 3942\, s_1^2 s_5 s_3^3 s_2 s_6^2 + 3942\, s_1 s_5^2 s_3^3 s_4 s_6 + 1020\, s_1^3 s_5 s_3^2 s_2^2 s_6^2 + 560\, s_1^4 s_5^2 s_3^2 s_4 s_6 +$
$160\, s_1^4 s_5^3 s_3 s_2 s_6 - 4464\, s_1 s_5 s_3^2 s_2^3 s_6^2 - 4464\, s_1 s_5 s_4^3 s_3^2 s_6 - 3750\, s_5^5 s_2 s_3 + 2250\, s_4 s_3^2 s_5^4 + 825\, s_3^2 s_2^2 s_5^4 - 1600\, s_4^3 s_5^3 s_3 +$
$16\, s_3^3 s_2^3 s_5^3 + 108\, s_3^4 s_2^3 s_6^2 + 16\, s_3^4 s_1^3 s_5^3 + 108\, s_3^5 s_1^3 s_6^2 + 1980\, s_6 s_3^2 s_1 s_5^3 s_2 - 682\, s_6 s_3 s_1^2 s_2^2 s_5^3 + 5832\, s_1 s_4^2 s_3^3 s_6^2 + 768\, s_1 s_4^5 s_3 s_6 -$
$192\, s_1 s_4^4 s_5^2 s_3 + 24\, s_3^2 s_2^4 s_5^2 s_6 + 16\, s_3^2 s_2^3 s_4^3 s_6 - 4\, s_3^2 s_2^3 s_4^2 s_5^2 - 27\, s_3^4 s_2^2 s_1^2 s_6^2 - 4\, s_3^3 s_2^2 s_1^2 s_5^3 + 2250\, s_3^2 s_2 s_1^4 s_6^3 + 6912\, s_3 s_2^4 s_1 s_6^3 -$
$72\, s_3^4 s_2 s_1 s_5^3 - 486\, s_3^5 s_2 s_1 s_6^2 + 768\, s_3 s_2^5 s_5 s_6^2 - 1600\, s_3 s_2^3 s_1^3 s_6^3 + 162\, s_3^4 s_2 s_5^2 s_6 + 16\, s_3^3 s_1^3 s_4^3 s_6 - 4\, s_3^3 s_1^3 s_4^2 s_5^2 - 108\, s_3^3 s_2^2 s_5^2 s_6 s_1 -$
$72\, s_3^3 s_2 s_4^3 s_1 s_6 + 18\, s_3^3 s_2 s_4^2 s_1 s_5^2 - 192\, s_3 s_2^4 s_1^2 s_5 s_6^2 - 6\, s_3^2 s_2^3 s_5^2 s_1^2 s_6 + s_3^2 s_2^2 s_4^2 s_1^2 s_5^2 + 324\, s_3^4 s_2 s_4 s_6 s_1 s_5 - 72\, s_3^3 s_2^3 s_4 s_5 s_6 -$
$4\, s_3^2 s_2^2 s_4^3 s_1^2 s_6 + 18\, s_3^3 s_2^2 s_4 s_1^2 s_6 s_5 + 24\, s_3^3 s_2 s_1^3 s_6 s_5^2 - 72\, s_3^4 s_1^3 s_4 s_6 s_5 + 5832\, s_5 s_3^3 s_2^2 s_6^2 + 6912\, s_5 s_4^4 s_3 s_6 - 1024\, s_4^6 s_6 +$
$256\, s_4^5 s_5^2 - 27\, s_2^4 s_5^4 s_1^2 + 256\, s_2^5 s_1^2 s_6^3 + 108\, s_2^5 s_5^4 - 1024\, s_2^6 s_6^3 - 486\, s_4 s_3^5 s_5 s_6 + 108\, s_4^3 s_3^4 s_6 - 27\, s_4^2 s_3^4 s_5^2 + 729\, s_3^6 s_6^2 +$
$108\, s_3^5 s_5^3 + 3125\, s_1^6 s_6^4$

Indication : $P = X^6 - s_1 X^5 + s_2 X^4 - s_3 X^3 + s_4 X^2 - s_5 X + s_6$. Si l’indication ne suffit pas, vous pourrez tourner la page mais pas avant de longues minutes de réflexion.

Plan de travail :

A) Relations entre les racines et les coefficients d'un polynôme de $\mathbb{C}[X]$.

B) Tout polynôme symétrique en plusieurs variables est un polynôme en les polynômes symétriques élémentaires. Les polynômes de Newton.

C) Le Résultant $R(P, Q)$ :

   (a) le système $UP + VQ = 0$ et la matrice de Sylvester,

   (b) la multiplication par $P$ modulo $Q$ et la multiplication par $Q$ modulo $P$,

   (c) l'expression du résultant avec les racines (deux méthodes).

D) Le discriminant $\Delta(P) = (-1)^{\frac{1}{2}n(n-1)}R(P, P')a_0^{-1}$ d'un polynôme $P = a_0X^n + \cdots + a_n$ de degré $n \geqslant 1$. [1] Expression avec les racines.

E) L'ambiguïté de $\prod_{i<j}(\alpha_i - \alpha_j)$. Le groupe alterné. Une (toute petite) ouverture vers la théorie de Galois.

Degré 4 : on prend $P = X^4 + rX^2 + sX + t$ (car on peut toujours annuler le terme en $X^3$ par une substitution $X \mapsto X + c$ qui décale les racines mais ne change pas le discriminant). La matrice de la multiplication par $P'$ modulo $P$ (dans la base canonique) est :

$$\begin{pmatrix} s & -4t & 0 & 2rt \\ 2r & -3s & -4t & 2rs \\ 0 & -2r & -3s & -4t + 2r^2 \\ 4 & 0 & -2r & -3s \end{pmatrix}$$

Le discriminant est son déterminant et il vaut

$$-4r^3s^2 - 27s^4 + 16r^4t - 128r^2t^2 + 144rts^2 + 256t^3$$

On peut constater que chaque contribution au développement du déterminant est de poids 12, avec $r$ de poids 2, $s$ de poids 3 et $t$ de poids 4. Car (le prouver par un raisonnement a priori) le poids de l'entrée $(i, j)$, $1 \leqslant i, j \leqslant 4$, est $3 + j - i$, comme résumé ci-dessous (le poids d'une entrée nulle est arbitraire) :

$$\begin{bmatrix} 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

et ceci implique (le prouver) que chaque contribution au développement du déterminant est a priori de poids $4 \times 3 = 12$. Sur la matrice de Sylvester pour le résultant (qui est de taille $7 \times 7$), c'est moins évident a priori que le déterminant sera de poids 12. Au recto de la feuille chaque monôme a poids $30 = 6 \times 5$.

1. on a donc $\Delta(P) = a_0^{2n-2}\Delta(a_0^{-1}P)$ avec $a_0$ le coefficient dominant de $P$.